Structure of a Morse form foliation on a closed surface in terms of genus

Irina Gelbukh

Current address: *CIC, IPN, 07738, DF, Mexico*

Abstract

We study the geometry of compact singular leaves γ and minimal components $\mathcal{C}^{min}$ of the foliation $\mathcal{F}_\omega$ of a Morse form ω on a genus g closed surface M_g^2 in terms of genus $g(*)$. We show that $c(\omega)+\sum_\gamma g(V(\gamma))+g(\bigcup\overline{\mathcal{C}^{min}})=g$, where $c(\omega)$ is the number of homologically independent compact leaves and $V(*)$ is a small closed tubular neighborhood. This allows us to prove a criterion for compactness of the singular foliation $\overline{\mathcal{F}}_\omega$, to estimate the number of its minimal components, and to give an upper bound on the rank $\operatorname{rk}\omega$, in terms of genus.

Keywords: Morse form foliation, minimal component, compact singular leave, genus, isotropic subgroup
2000 MSC: 57R30, 58K65

1. Introduction and statement of main results

Let ω be a Morse form, i.e., a closed 1-form with Morse singularities—locally the differential of a Morse function, on a genus g closed surface $M=M_g^2$. It defines a foliation $\mathcal{F}_\omega$ on $M\setminus\operatorname{Sing}\omega$ and a singular foliation $\overline{\mathcal{F}}_\omega$ (with possible singular leaves) on the whole M. A leaf $\gamma\in\mathcal{F}_\omega$ is compactifiable if $\gamma\cup\operatorname{Sing}\omega$ is compact; then the closure $\overline{\gamma}$ is a circle or a segment.

Compact leaves of $\mathcal{F}_\omega$ are circles; connected components of their union are cylinders called maximal components $\mathcal{C}_i^{max}$. Non-compactifiable leaves form minimal components $\mathcal{C}_j^{min}$; each such leaf is dense in its minimal component [9, 13]. The number of maximal and minimal components is finite. Obviously, all components are mutually disjoint and

$$M=\bigcup\overline{\mathcal{C}_i^{max}}\cup\bigcup\overline{\mathcal{C}_j^{min}}.$$

The set $\bigcup\partial\mathcal{C}_i^{max}$ consists of a finite number of compactifiable leaves and singularities. The set of its connected components coincides with the set of all compact singular leaves.

We study interrelation of some characteristics of maximal and minimal components. Denote by $c(\omega)$ be the number of homologically independent compact leaves of $\mathcal{F}_\omega$ and by $m(\omega)$ the number of minimal components. Arnoux and Levitt [2] and Maier [14] have shown that $m(\omega)\le g$. Later it was shown [4] that

$$c(\omega)+m(\omega)\le g; \tag{1}$$

moreover, if the form is weakly generic (each connected component of $\bigcup\partial\mathcal{C}_i^{max}$ contains a unique singularity) then [7]

$$c(\omega)+m(\omega)=g-\frac{k(\omega)}{2}, \tag{2}$$

where $k(\omega)=|\operatorname{Sing}\omega\cap\bigcup\operatorname{int}(\overline{\mathcal{C}^{min}})|$ is the number of singularities $s\in\operatorname{Sing}\omega$ "inside" minimal components.

Email address: <last name>@member.ams.org (Irina Gelbukh)
URL: www.i.gelbukh.com (Irina Gelbukh)

Preprint submitted to Elsevier *June 8, 2011*

Cite this paper as: *Differential Geometry and its Applications* 29:473–492, 2011.
Pre-print version. Final version: http://dx.doi.org/10.1016/j.difgeo.2011.04.029

Levitt [12], Aranson and Zhuzhoma [1], and Kono [11] have studied the structure of quasiminimal components (which for Morse forms coincide with $\overline{\mathcal{C}_j^{min}}$) in terms of leaves and singularities. In contrast, we address their genus. Namely, we show that the genus $g(*)$ of various structural elements of the foliation is useful for characterization of its topology. Our main result (Theorem 42) is

$$c(\omega) + \sum_\gamma g\left(V\left(\gamma\right)\right) + g\left(\bigcup \overline{\mathcal{C}_j^{min}}\right) = g, \tag{3}$$

where γ are all compact singular leaves and $V(*)$ is a small closed tubular neighborhood.

This improves on (1) since the last summand is at least $m(\omega)$ (Corollary 10), which gives (Corollary 44)

$$c(\omega) + m(\omega) \leq g - \sum_\gamma g(V(\gamma)). \tag{4}$$

Equation (3) also generalizes (2) to a wider class of forms, since $k(\omega)$ reflects the genus of minimal components (Theorem 50, Corollary 51). In addition, it generalizes the result of Zorich [18], who showed that for a generic form (each singular leaf contains a unique singularity) with maximal rank (the rank of the group of periods), $\operatorname{rk}\omega = 2g$, it holds

$$\sum g(\overline{\mathcal{C}_j^{min}}) = g. \tag{5}$$

Indeed, in this special case the two first summands of (3) are zero (Proposition 26), and then (5) follows from (3) by Lemma 9.

Finally, (3) gives a criterion for compactness of a foliation (Theorem 43): $\overline{\mathcal{F}}_\omega$ is compact (all leaves are compact) iff

$$c(\omega) + \sum_\gamma g(V(\gamma)) = g,$$

which improves on the condition—criterion if ω is generic—for compactness of $\overline{\mathcal{F}}_\omega$ given by Mel'nikova [15]: if $c(\omega) = g$ then $\overline{\mathcal{F}}_\omega$ is compact. In particular, if $\sum_\gamma g(V(\gamma)) = g$ then $\overline{\mathcal{F}}_\omega$ is compact and all its non-singular leaves are homologically trivial; moreover, the form is exact: $\omega = df$ (Proposition 26).

While $g(V(\gamma))$ depends on the embedding of γ in M_g^2, in some cases we can tell that $g(V(\gamma)) > 0$ solely on the basis of the structure of γ. Indeed, consider γ as a graph. The genus $g(\gamma)$ of a graph is defined as the minimal integer k such that the graph can be embedded in a surface M_k^2; cf. Figure 8. Obviously, $g(\gamma) \leq g(V(\gamma))$; i.e., the structure of a compact singular leaf considered as a graph can give useful information about the foliation structure (Corollary 35).

In particular, most of our inequalities still hold, and equalities turn into inequalities, in terms of $\sum_\gamma g(\gamma)$ that is independent of the embedding. For example, (4) rewritten as

$$c(\omega) + m(\omega) \leq g - \sum_\gamma g(\gamma)$$

still improves on (1).

Since for any leaf γ it holds $\int_\gamma \omega = 0$, the structure of compact elements of the foliation defines zero periods of the form ω, which affects its rank (Proposition 26):

$$\operatorname{rk}\omega + c(\omega) \leq 2\left(g - \sum_\tau g(V(\tau))\right), \tag{6}$$

where each τ is a connected component of $\bigcup \partial\mathcal{C}_i^{max}$; τ is either a compact singular leaf γ or a boundary component δ of the set $\bigcup_j \overline{\mathcal{C}_j^{min}}$, i.e. $\{\tau\} = \{\gamma\} \cup \{\delta\}$.

For compact $\overline{\mathcal{F}}_\omega$ we have (Corollary 28):

$$\operatorname{rk}\omega \leq g - \sum g(V(\gamma)).$$

This improves the result of Mel'nikova [16], who proved that $\operatorname{rk}\omega \leq g$.

We consider in detail a class of Morse forms for which compact singular leaves give sufficient information for (6), i.e., for which $\sum g(V(\delta)) = 0$. Namely, we introduce a class of very weakly generic forms: those for which each δ contains a unique singularity and is thus S^1. This class generalizes the classes of generic forms and weakly generic forms [7].

Minimal components of a very weakly generic form are non-adjacent ($\overline{\mathcal{C}_i^{min}} \cap \overline{\mathcal{C}_j^{min}} = \emptyset$), so (Lemma 9)

$$g\left(\bigcup \overline{\mathcal{C}_j^{min}}\right) = \sum g(\overline{\mathcal{C}_j^{min}})$$

and (3) becomes

$$c(\omega) + \sum_{\gamma} g(V(\gamma)) + \sum g(\overline{\mathcal{C}_j^{min}}) = g.$$

Moreover, for a very weakly generic form the genus $g(\overline{\mathcal{C}_j^{min}})$ is defined by the number $k_j = |\operatorname{Sing}\omega \cap \operatorname{int}(\overline{\mathcal{C}_j^{min}})|$ of singularities inside $\mathcal{C}_j^{min}$ (Theorem 47):

$$g(\overline{\mathcal{C}_j^{min}}) = 1 + \frac{k_j}{2}.$$

This further rewrites (3) as (Theorem 50)

$$c(\omega) + m(\omega) + \frac{k(\omega)}{2} + \sum_{\gamma} g(V(\gamma)) = g, \tag{7}$$

where is the total number of singularities inside minimal components. Some properties of $k(\omega)$ are given in Proposition 49.

For a very weakly generic form the latter equality defines, in particular, the number $m(\omega)$ of minimal components—a problem that has received attention in the past [2, 4, 7, 14]. Given the difficulty of exact calculation of $g(V(\gamma))$, we also give some simplified estimations of $m(\omega)$. Note that while (1) is a simple *upper* bound on $m(\omega)$, we are not aware of any *lower* bound on $m(\omega)$ existing in the literature.

Consider $\ker[\omega] = \langle z \in H_1(M) \mid \int_z \omega = 0 \rangle$ and the rank $h(\ker[\omega])$ of its maximal isotropic subgroup (subgroup consisting of non-intersecting cycles); it is calculated in Lemma 14. Equation (7) implies (Theorem 53)

$$g - \frac{k(\omega)}{2} - h(\ker[\omega]) \leq m(\omega) \leq g - \frac{k(\omega)}{2} - c(\omega)$$

with $m(\omega) > 0$ if $k(\omega) > 0$; for given g, $k(\omega)$, and $h(\ker[\omega])$ the bounds are exact and all intermediate values are reached.

Since studying the structure of $\ker[\omega]$ can also be difficult, we give a weaker lower bound not involving $h(\ker[\omega])$ (Corollary 54):

$$m(\omega) \geq \operatorname{rk}\omega - g - \frac{k(\omega)}{2}.$$

This bound is efficient only for large $\operatorname{rk}\omega$, specifically, for $\operatorname{rk}\omega \geq g$. However, a "typical" Morse form (in terms of measure) has $\operatorname{rk}\omega = 2g$.

Finally we build an example that shows that our system of relations between g, $m(\omega)$, $c(\omega)$, $h(\ker[\omega])$, and $k(\omega)$ is complete: all combinations of their values allowed by our inequalities are reached even in the class of very weakly forms; in particular, the corresponding bounds are exact.

The paper is organized as follows. In Section 2 we introduce some necessary definitions and facts concerning a Morse form foliation and prove some useful lemmas. In Section 3 we study some properties of isotropic subgroups associated with the foliation; their geometric interpretation is given in Section 4, where we analyze the topology of $\bigcup \partial\mathcal{C}_i^{max}$. In Section 5 we discuss some properties of minimal components. In Section 6 we prove our main theorem (3). In Section 7 we study minimal components of very weakly generic forms and give the estimates on $m(\omega)$. Finally, in Section 8 we show completeness of our characterization and in particular exactness of our bounds.

2. Definitions and basic facts

Let us introduce, for future reference, some necessary notions and facts about Morse forms and their foliations. By $M = M_g^2$ we denote a genus g closed orientable surface.

2.1. Morse form

A closed 1-form on M is called a *Morse form* if it is locally the differential of a Morse function. Let ω be a Morse form and $\operatorname{Sing}\omega = \{p \in M \mid \omega(p) = 0\}$ be the set of its singularities; this set is finite since the singularities are isolated and M is compact.

By the Morse lemma, in a neighborhood of $s \in \operatorname{Sing}\omega$ on M_g^2 there exist local coordinates (x^1, x^2) such that $\omega(x) = \pm x^1 dx^1 + x^2 dx^2$. If the sign is positive then s is a *center*, otherwise it is a *conic singularity.* We denote the set of centers by Ω_0 and the set of conic singularities by Ω_1, so that $\operatorname{Sing}\omega = \Omega_0 \cup \Omega_1$. By the Poincaré—Hopf theorem, it holds

$$|\Omega_1| - |\Omega_0| = 2g - 2. \tag{8}$$

The *rank* of a closed 1-form ω is the rank of its group of periods:

$$\operatorname{rk}\omega = \operatorname{rk}_{\mathbb{Q}} \left\{ \int_{z_1} \omega, \ldots, \int_{z_{2g}} \omega \right\},$$

where $z_1, \ldots, z_{2g}$ is a basis of $H_1(M_g^2)$. For an exact form, $\operatorname{rk}\omega = 0$.

2.2. Morse form foliation

On $M \setminus \operatorname{Sing}\omega$, the form ω defines a (non-singular) foliation $\mathcal{F}_\omega$. A leaf $\gamma \in \mathcal{F}_\omega$ is *compactifiable* if $\gamma \cup \operatorname{Sing}\omega$ is compact (thus compact leaves are compactifiable); otherwise it is *non-compactifiable.* If a foliation contains only compactifiable leaves then it is called *compactifiable.*

Lemma 1 ([7]). *Let $\gamma^0 \in \mathcal{F}_\omega$ be a non-compact compactifiable leaf such that $\gamma^0 \cup s$ is compact for some $s \in \operatorname{Sing}\omega$. Then in any neighborhood of $\overline{\gamma^0} = \gamma^0 \cup s$ there exists a compact leaf $\gamma \in \mathcal{F}_\omega$.*

The foliation $\mathcal{F}_\omega$ defines a decomposition of M_g^2 into mutually disjoint sets defined below; see Figure 1 [6]:

$$M_g^2 = \left(\bigcup \mathcal{C}_i^{max}\right) \cup \left(\bigcup \mathcal{C}_j^{min}\right) \cup \left(\bigcup \gamma_k^0\right) \cup \operatorname{Sing}\omega. \tag{9}$$

Maximal components $\mathcal{C}_i^{max}$ are connected components of the union of all compact leaves. On M_g^2 the notion of maximal component coincides with the notion of periodic component [14]. Unless $\operatorname{Sing}\omega = \emptyset$, each maximal component is a cylinder over a compact leaf: $\mathcal{C}_i^{max} \cong \gamma_i \times (0, 1)$. Consider the group $H_\omega \subseteq H_1(M_g^2)$ generated by the homology classes of all compact leaves: $H_\omega = \langle [\gamma_i], \gamma_i \in \mathcal{F}_\omega \rangle$ [4]; $c(\omega) = \operatorname{rk} H_\omega$ denotes the number of homologically independent compact leaves.

Minimal components $\mathcal{C}_i^{min}$ of the foliation are connected components of the union of all non-compactifiable leaves. A foliation that has exactly one minimal component and no maximal components is called *minimal.* Each non-compactifiable leaf is dense in its minimal component [2, 9]. We denote by $m(\omega)$ the number of minimal components.

Components $\mathcal{C}_i^{max}$ and $\mathcal{C}_i^{min}$ are open; their boundaries lie in the union $(\bigcup_k \gamma_k^0) \cup \operatorname{Sing}\omega$ of all non-compact compactifiable leaves and singularities. The number of components, as well as the number of non-compact compactifiable leaves γ_k^0, is finite.

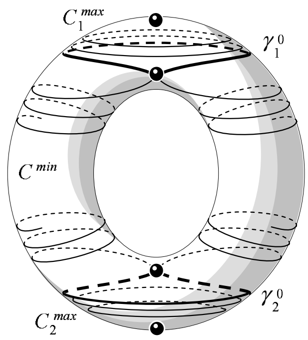

Figure 1: Decomposition of T^2: $\mathcal{C}_i^{max}$ are maximal components, they are cylinders; $\mathcal{C}^{min}$ is a minimal component, it is a torus with two holes; the components are connected by compactifiable leaves γ_i^0 and singularities.

In homology terms, decomposition (9) implies [4]:

$$H_1(M_g^2) = \langle DH_\omega,\, i_* H_1\left(\bigcup \partial \mathcal{C}_i^{max}\right),\, j_* H_1\left(\bigcup \overline{\mathcal{C}_j^{min}}\right) \rangle, \tag{10}$$

where D is a Poincaré duality map and i, j are the inclusion maps.

2.3. Singular foliation

While a foliation $\mathcal{F}_\omega$ is defined on $M \setminus \operatorname{Sing}\omega$, a *singular foliation* $\overline{\mathcal{F}_\omega}$ is an equivalence relation defined on the whole M: two points $p, q \in M$ belong to the same *leaf* of $\overline{\mathcal{F}_\omega}$ if there exists a path $\alpha : [0,1] \to M$ with $\alpha(0) = p$, $\alpha(1) = q$ and $\omega(\dot\alpha(t)) = 0$ for all t [3]. A *singular leaf* contains a singularity.

The singular foliation $\overline{\mathcal{F}_\omega}$ differs from $\mathcal{F}_\omega$ only by possibly merging together some its leaves: indeed, non-singular leaves of $\overline{\mathcal{F}_\omega}$ are leaves of $\mathcal{F}_\omega$; the number of singular leaves of $\overline{\mathcal{F}_\omega}$ is finite, and each such leaf consists of a finite number of non-compact leaves of $\mathcal{F}_\omega$ and singularities. $\mathcal{F}_\omega$ is compactifiable iff $\overline{\mathcal{F}_\omega}$ is compact, i.e., all its leaves are compact.

2.4. Very weakly generic forms

Definition 2 ([3]). A Morse form is called *generic* if each its singular leaf contains a unique singularity.

On M_g^2 this is equivalent to the requirement for each non-compact compactifiable leaf of $\mathcal{F}_\omega$ to be compactified by only one singularity, i.e., for its closure to be a circle (as γ_i^0 in Figure 1) and not a segment (as γ^0 in Figure 2 (a)). In particular, compact singular leaves of a generic form are figures of eight, as τ_1 in Figure 6.

Generic forms are "typical" Morse forms in the sense that their set is open and dense in the space of Morse forms [3]. A reader only interested in generic forms may skip the next two definitions, since all our results are applicable to generic forms.

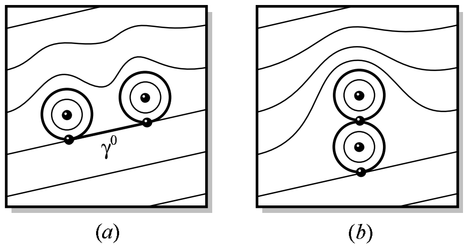

Figure 2: Foliations on T^2 with one minimal component. The form (a) is weakly generic, though not generic; the form (b) is not even very weakly generic.

Definition 3. A Morse form is *weakly generic* if any connected component $\partial_j\mathcal{C}_i$ of the boundary of any its (minimal or maximal) component $\mathcal{C}_i$ contains a unique singularity.

On M_g^2, this means that only those non-compact compactifiable leaves of $\mathcal{F}_\omega$ that lie outside minimal components are required to be compactified by only one singularity, while those inside minimal components can form segments; see Figure 2. *Par abus de langage* we say that a leaf or singularity is *inside* a component $\mathcal{C}$ if it belongs to $\operatorname{int}(\overline{\mathcal{C}})$. In other words, a weakly generic form is a form that is generic outside minimal components; in particular, all compact singular leaves of a weakly generic form are figures of eight.

Definition 4. We call a Morse form *very weakly generic* if any connected component $\partial_j\mathcal{C}^{min}$ of the boundary of any its *minimal* component contains a unique singularity.

On M_g^2 this means that only those leaves that lie on the boundary of minimal components are required to be compactified by only one singularity, i.e., each $\partial_j\mathcal{C}^{min}$ is either a circle $\gamma^0 \cup s$ or a single $s \in \underline{\operatorname{Sing}\omega}$ inside $\mathcal{C}^{min}$; the former are connected components of $\partial\overline{\mathcal{C}^{min}}$. Compact singular leaves of a very weakly generic form do not have to be figures of eight.

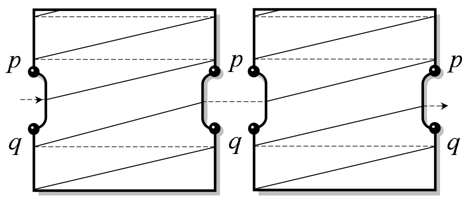

Figure 3: Minimal foliation on $M_g^2 = T^2 \sharp T^2$; cf. Figure 9. Figure adapted from [5].

Lemma 5. *If ω is very weakly generic then $\overline{\mathcal{C}_i^{min}} \cap \overline{\mathcal{C}_j^{min}} = \emptyset$.*

Proof. Connected components of $\partial\overline{\mathcal{C}^{min}}$ are circles $\overline{\gamma^0}$. Out of local considerations, each such circle separates the $\mathcal{C}^{min}$ from not more than one another component, which by Lemma 1 must be a maximal component and thus cannot be another minimal component. $\square$

For very weakly generic forms, the topology of minimal components is tightly connected with the singularities inside minimal components. Let $k_i = |\operatorname{int}(\overline{\mathcal{C}_i^{min}}) \cap \operatorname{Sing}\omega|$ be the number of singularities inside $\mathcal{C}_i^{min}$; we denote by $k(\omega) = \sum_{i=1}^{m(\omega)} k_i$ the total number of singularities inside all minimal components. In Figure 3, $k(\omega) = 2$. In fact, our results hold for an even wider class of forms, such as the one shown in Figure 2 (b), but not in Figure 9; however, we will treat such generalizations in a separate paper.

2.5. The genus of a surface

Definition 6. The *genus* $g(S)$ of an orientable surface S is the maximum number of cuttings along closed simple curves without increasing the number of its connected components.

For closed surfaces, $g(M_g^2) = g$. Let $S \subseteq M_g^2$ be a closed subset, obviously, $g(S) \le g$.

Lemma 7. *Let $i : S \hookrightarrow M_g^2$ be a surface with boundary, $i_* : H_1(S) \to H_1(M_g^2)$. Then* $\ker i_* = 0$.

PROOF. Denote $M = M_g^2$. Consider the long exact sequence of the pair (M, S):

$$\dots \to H_2(M) \xrightarrow{j} H_2(M, S) \xrightarrow{\partial *} H_1(S) \xrightarrow{i*} H_1(M) \to \dots$$

Since $H_2(M) = H_2(M, S) = \mathbb{Z}$ and $\operatorname{im} j = \ker \partial_* = \mathbb{Z}$, we have $\operatorname{im} \partial_* = 0 = \ker i_*$. □

Corollary 8. *The closed curves $\alpha_1, \dots, \alpha_{g(S)}$ from Definition 6 for $S \subset M_g^2$ are homologically independent in $H_1(M_g^2)$.*

Let us consider some subsets of M_g^2 covered by minimal components:

Lemma 9. *For a Morse form foliation it holds*
(i) $g(\overline{\mathcal{C}^{min}}) \ge 1$,
(ii) $g\left(\bigcup_j \overline{\mathcal{C}_j^{min}}\right) \ge \sum_j g(\overline{\mathcal{C}_j^{min}})$.

For a very weakly generic form, the latter turns into equality $g\left(\bigcup_j \overline{\mathcal{C}_j^{min}}\right) = \sum_j g(\overline{\mathcal{C}_j^{min}})$.

PROOF. (i) A subset $\overline{\mathcal{C}^{min}} \subseteq M_g^2$ admits a flow having a dense orbit (transitive flow), so it is connected and $g(\overline{\mathcal{C}^{min}}) \ne 0$ [10].

(ii) Consider two minimal components, $\mathcal{C}_1 = \mathcal{C}_1^{min}$ and $\mathcal{C}_2 = \mathcal{C}_2^{min}$. If $\overline{\mathcal{C}_1} \cap \overline{\mathcal{C}_2} = \emptyset$ then, obviously, $g(\overline{\mathcal{C}_1} \cup \overline{\mathcal{C}_2}) = g(\overline{\mathcal{C}_1}) + g(\overline{\mathcal{C}_2})$. In particular, by Lemma 5 this holds for all minimal components of a very weakly generic form. Now let $\overline{\mathcal{C}_1} \cap \overline{\mathcal{C}_2} \ne \emptyset$; then $\overline{\mathcal{C}_1} \cap \overline{\mathcal{C}_2} \subset \partial\mathcal{C}_1 \cup \partial\mathcal{C}_2$, so by definition, $g(\overline{\mathcal{C}_1}) + g(\overline{\mathcal{C}_2}) \le g(\overline{\mathcal{C}_1} \cup \overline{\mathcal{C}_2})$. Induction on the number of minimal components completes the proof. □

Corollary 10. $g\left(\bigcup_{j=1}^{m(\omega)} \overline{\mathcal{C}_j^{min}}\right) \ge m(\omega)$.

For a Morse form that is not very weakly generic a strict inequality can hold:

Example 11. Consider a foliation on M_3^2 covered by two minimal components $\mathcal{C}_j^{min}$ as in Figure 4. We have $g(\overline{\mathcal{C}_1^{min}}) = g(\overline{\mathcal{C}_2^{min}}) = 1$ and $g(\bigcup \overline{\mathcal{C}_j^{min}}) = 3$, so $\sum_j g(\overline{\mathcal{C}_j^{min}}) < g(\bigcup \overline{\mathcal{C}_j^{min}})$.

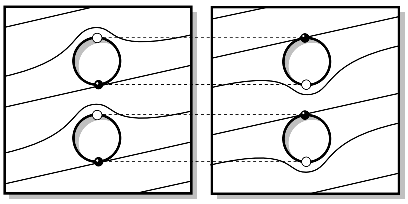

Figure 4: Two tori with a winding ($\mathcal{C}_j^{min}$) are connected by means of two holes, forming an M_3^2.

3. On some maximal isotropic subgroups in $H_1(M)$

A singular foliation $\overline{\mathcal{F}}_\omega$ has three types of leaves: compact non-singular leaves, compact singular leaves, and non-compact leaves. We consider their geometric characteristics: isotropic subgroups generated by leaves (this section) and the genus of a neighborhood of a leaf (below).

3.1. Intersection of cycles and isotropic subgroups

Consider on $H_1(M_g^2)$ the intersection of cycles

$$\cdot : H_1(M_g^2) \times H_1(M_g^2) \to \mathbb{Z};$$

it is skew-symmetric and non-degenerated.

Definition 12. A subgroup $H \subset H_1(M_g^2)$ is called *isotropic* with respect to the cycle intersection $\cdot$ if for any $z, z' \in H$ it holds $z \cdot z' = 0$.

Obviously, for an isotropic subgroup $H \subseteq H_1(M_g^2)$, it holds $\operatorname{rk} H \leq g$.

Definition 13. *Isotropic rank* $h(G)$ of a subgroup $G \subseteq H_1(M_g^2)$ is the rank of a maximal isotropic subgroup $H \subseteq G$: $h(G) = \operatorname{rk} H$.

For M_g^2 (unlike higher-dimensional case) isotropic rank is well-defined because $\operatorname{rk} H$ does not depend on the choice of H:

Lemma 14 ([7]). *Let $G \subseteq H_1(M_g^2)$. Then*

$$h(G) = \operatorname{rk} G - \frac{1}{2} \operatorname{rk} \|z_i \cdot z_j\|,$$

where $\{z_i\}$ is any basis of G.

Corollary 15. *For $G \subseteq H_1(M_g^2)$, it holds $h(G) \geq \frac{1}{2} \operatorname{rk} G$.*

The value $h\left(H_{n-1}(M)\right)$, $n = \dim M$, properly generalized, is an important topological invariant of a manifold denoted by $h(M)$ [4, 16, 17]; specifically,

$$h\left(H_1(M_g^2)\right) = h(M) = g. \tag{11}$$

For a surface $S \subset M_g^2$ we denote $h(S) = h(H_1(S))$; the isotropic rank does not depend on the inclusion:

Lemma 16. *Let $i : S \hookrightarrow M_g^2$ be a surface with boundary and $G \subseteq H_1(S)$ be a maximal isotropic subgroup. Then i_*G is a maximal isotropic subgroup in $i_*H_1(S) \subseteq H_1(M_g^2)$ and $\operatorname{rk}(i_*G) = \operatorname{rk} G$, i.e. $h(i(S)) = h(S)$.*

PROOF. Obviously, $i_*G \subset i_*H_1(S)$ is a maximal isotropic subgroup. By Lemma 7, $\ker i_* = 0$, so $\operatorname{rk}(i_*G) = \operatorname{rk} G$. □

Let us consider an important example of a maximal isotropic subgroup on a surface:

Proposition 17. *Let S be a compact orientable surface with boundary $\partial S = \bigcup \partial_j$, $\partial_j = S^1$, and α_i, $i = 1, \ldots, g(S)$, be simple closed curves from Definition 6 that define its genus. Then*

$$H = \langle [\alpha_1], \ldots, [\alpha_g], [\partial_1], \ldots, [\partial_l] \rangle$$

is a maximal isotropic subgroup and

$$h(H_1(S)) = g(S) + \operatorname{rk}\langle [\partial_j] \rangle.$$

In addition, there exist non-intersecting curves $\beta_1, \ldots, \beta_g \subset S$ such that $[\alpha_i] \cdot [\beta_j] = \delta_{ij}$ and $\partial_i \cap \beta_j = \emptyset$. The cycles $[\alpha_i], [\beta_j], [\partial_k] \in H_1(S)$ are related in the following way:

$$\operatorname{rk}\langle [\alpha_i], [\beta_j], [\partial_k] \rangle = 2g(S) + \operatorname{rk}\langle [\partial_k] \rangle.$$

This proposition generalizes (11) and allows one to study submanifolds.

Proof. Since $\alpha_i \cap \alpha_j = \alpha_i \cap \partial_j = \partial_i \cap \partial_j = \emptyset$, the subgroup H is isotropic. Let us show that it is maximal. Consider a cycle $z \in H_1(S)$ such that $z \cdot H = 0$. Realize z by a curve α, $\alpha \cap \alpha_i = \emptyset$. Denote by S' the result of cutting S open along all α_i. By construction, S' is a sphere with holes and $\alpha \subset S'$. So any connected component of α splits S' up and thus is induced from $\partial S' = \partial S \cup \left(\bigcup(\alpha_i^+ \cup \alpha_i^-)\right)$, where $\alpha_i^\pm$ are two copies of α_i. This implies $z = [\alpha] \in \langle[\alpha_i], [\partial_j]\rangle = H$.

By the choice of α_i, we have $H = \langle[\alpha_1]\rangle \oplus \cdots \oplus \langle[\alpha_g]\rangle \oplus \langle[\partial_1], \ldots, [\partial_l]\rangle$ and $[\alpha_i] \neq 0$, which gives $\operatorname{rk} H = g + \operatorname{rk}\langle[\partial_j]\rangle$.

Gluing up S by disks we obtain M_g^2, where the desired curves β_i exist; without loss of generality we can suppose $\beta_i \subset S$.

Now let $z \in \langle[\alpha_i], [\beta_j]\rangle \cap \langle[\partial_k]\rangle$; then

$$z = \sum_i n_i^\alpha [\alpha_i] + \sum_j n_j^\beta [\beta_j] = \sum_k m_k [\partial_k].$$

Since $\beta_l \cap \alpha_i = \emptyset$, we obtain $z \cdot [\beta_l] = 0$, so $n_l^\alpha = 0$; similarly, all $n_l^\beta = 0$, i.e. $z = 0$. Thus $\langle[\alpha_i], [\beta_j], [\partial_k]\rangle = \langle[\alpha_i], [\beta_j]\rangle \oplus \langle[\partial_k]\rangle$. Obviously, $\langle[\alpha_i], [\beta_j]\rangle = \langle[\alpha_i]\rangle \oplus \langle[\beta_j]\rangle$ and $\operatorname{rk}\langle[\alpha_i]\rangle = \operatorname{rk}\langle[\beta_j]\rangle = g(S)$. This implies $\operatorname{rk}\langle[\alpha_i], [\beta_j], [\partial_k]\rangle = 2g(S) + \operatorname{rk}\langle[\partial_k]\rangle$. □

Note that

$$\operatorname{rk}\langle[\partial_j]\rangle = \#(\partial S) - \#'(S),$$

where $\#(*)$ is the number of connected components and $\#'(*)$ is the number of connected components with non-empty boundary, and thus

$$h(H_1(S)) = g(S) + \#(\partial S) - \#'(S).$$

3.2. Isotropic subgroups associated with the foliation

Since leaves of a foliation do not intersect, isotropic subgroups related with the foliation can be used to describe its geometrical structure.

Compact leaves generate an isotropic subgroup $H_\omega \subset H_1(M_g^2)$; denote $c(\omega) = \operatorname{rk} H_\omega$.

Compact singular leaves correspond to closed curves lying in $\bigcup \partial\mathcal{C}_j^{max}$; see (9). These closed curves generate a subgroup $G_c = i_* H_1(\bigcup \partial\mathcal{C}_j^{max})$; i is the inclusion map. The subgroup G_c is not necessarily isotropic (see Figure 5); though for a weakly generic form it is (see Lemma 18 below). Since any compact leaf in $\mathcal{C}_j^{max}$ is induced from $\partial\mathcal{C}_j^{max}$, we have $H_\omega \subseteq G_c$.

Note that for any closed curve γ lying in a leaf of the foliation, it holds $\int_\gamma \omega = 0$. Thus

$$H_\omega \subseteq G_c \subseteq \ker[\omega], \tag{12}$$

where $\ker[\omega] = \{z \in H_1(M_g^2) \mid \int_z \omega = 0\}$. The subgroup $\ker[\omega]$ can be isotropic, but in general it is not. Consider isotropic ranks of the groups from (12), then

$$c(\omega) \leq h_c(\omega) \leq h(\ker[\omega]) \leq g. \tag{13}$$

where $h_c(\omega) = h(G_c)$ is the number of non-intersecting cycles lying in the boundaries of maximal components—some of them are homologous to compact leaves; recall that $c(\omega)$ is the number of homologically independent compact leaves of $\mathcal{F}_\omega$ and $h(\ker[\omega])$ is the number of non-intersecting cycles with zero integral.

These three numbers characterize the geometrical structure of $\mathcal{F}_\omega$. In the remainder of this section we will study the inequality (13).

If $\operatorname{rk}\omega = 2g$, we have $\ker[\omega] = 0$, so $h(\ker[\omega]) = h_c(\omega) = c(\omega) = 0$. This is a trivial case: if $g \neq 0$ the foliation consists of minimal components and (optionally) homologically trivial compact leaves.

Consider the lower bound in (13).

Lemma 18. *For a weakly generic form, $c(\omega) = h_c(\omega)$.*

Proof. For a weakly generic form we have $G_c \subseteq H_\omega$. Indeed, each connected component γ of $\bigcup_j \partial\mathcal{C}_j^{max}$ is O-shaped or 8-shaped; Lemma 1 gives $i_* H_1(\gamma) \subseteq H_\omega$. □

The condition for ω to be weakly generic is important: Figure 5 shows the case $0 = c(\omega) \neq h_c(\omega) = 1$. In the next section we will discuss the difference $h_c(\omega) - c(\omega)$ and its geometric meaning.

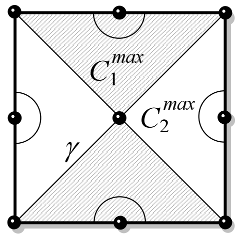

Figure 5: Decomposition of T^2 contains two $\mathcal{C}_i^{max}$ and a unique compact singular leaf γ that glues them together. In this case the subgroup $G_c = i_* H_1(\gamma)$ coincides with the whole group $H_1(M_g^2)$.

Now consider the upper bound in (13). By Lemma 14 and since $\operatorname{rk}\ker[\omega] = 2g - \operatorname{rk}\omega$, for $h(\ker[\omega])$ we have

$$g - \operatorname{rk}\omega \leq h(\ker[\omega]) \leq \min(2g - \operatorname{rk}\omega, g).$$

In particular,

$$0 \leq h(\ker[\omega]) \leq g. \tag{14}$$

If $\operatorname{rk}\omega = 0$, we have $h(\ker[\omega]) = g$. In fact this holds for any compactifiable foliation (see Corollary 41 below).

Denote $S = \bigcup \overline{\mathcal{C}_i^{min}}$ the minimal part of the foliation; let $j : S \hookrightarrow M_g^2$ be the inclusion map.

Lemma 19. *If $\ker[\omega] \cap j_* H_1(S) \subseteq G_c$ then $h_c(\omega) = h(\ker[\omega])$.*

Proof. Rewrite (10) as

$$H_1(M) = \langle DH_\omega,\, G_c,\, j_* H_1(S)\rangle. \tag{15}$$

Let $\ker[\omega] \cap j_* H_1(S) \subseteq G_c$. Consider a maximal isotropic subgroup $H \subseteq \ker[\omega]$, $H \supseteq H_\omega$; then $H \subseteq \langle G_c,\, j_* H_1(S)\rangle \cap \ker[\omega] \subseteq G_c$, which with (13) gives $h(\ker[\omega]) = \operatorname{rk} H = h_c(\omega)$. □

Geometrically this means that $\operatorname{rk}\omega$ is maximal in the set $S = \bigcup_j \overline{\mathcal{C}_j^{min}}$:

Lemma 20. $\ker[\omega] \cap j_* H_1(S) \subseteq G_c$ *iff* $\operatorname{rk}\omega|_S = 2g(S)$.

Proof. Consider the exact sequence of a pair:

$$\longrightarrow H_1(\partial S) \xrightarrow{\alpha_*} H_1(S) \xrightarrow{\beta_*} H_1(S, \partial S) \longrightarrow .$$

Then $H_1(S) = \alpha_* H_1(\partial S) \oplus \beta_* H_1(S)$, where $\beta_* H_1(S)$ contains only the cycles not induced from ∂S. Since $\partial S \subseteq \bigcup \partial\mathcal{C}_i^{max}$, we obtain $j_*\alpha_* H_1(\partial S) \subseteq G_c \subseteq \ker[\omega]$. Thus

$$\ker[\omega] \cap j_* H_1(S) = j_*\alpha_* H_1(\partial S) \oplus \left(\ker[\omega] \cap j_*\beta_* H_1(S)\right).$$

So the condition $\ker[\omega] \cap j_* H_1(S) \subseteq G_c$ is equivalent to $\ker[\omega] \cap j_*\beta_*(H_1(S)) = 0$, i.e., $\operatorname{rk}\omega|_S = \operatorname{rk}\beta_*(H_1(S)) = 2g(S)$. □

Corollary 21. *If $\mathcal{F}_\omega$ is compactifiable then $h_c(\omega) = h(\ker[\omega])$.*

4. Topology of the compact part of the foliation

Let $M_g^2 = C \cup S$, where $C = \bigcup \overline{\mathcal{C}_i^{max}}$ is the compact part of the foliation and $S = \bigcup \overline{\mathcal{C}_i^{min}}$ is its minimal part; $C \cap S = \partial C = \partial S$. Note that the boundary may contain singularities. In the previous section we have briefly discussed the minimal part S; in this section we study the compact part C.

Consider $\bigcup \partial\mathcal{C}_j^{max} \subset C$. Denote by τ a connected component of $\bigcup \partial\mathcal{C}_j^{max}$. If $\tau \subset \operatorname{int}(C)$, it is a compact singular leaf, as τ_1 in Figure 6; it can also be a part of a non-compact singular leaf consisting of compactifiable leaves and the corresponding singularities, as τ_2 in Figure 6.

The boundary of C may contain singularities, as in Figure 6 (a), so we consider its small closed neighborhood $\widetilde{C}$, as in Figure 6 (b). The boundary of $\widetilde{C}$ is non-singular, it consists of non-intersecting circles. In addition, the homology groups of C and $\widetilde{C}$ are isomorphic, $H_1(C) = H_1(\widetilde{C})$. Denote by $V(\tau)$ a small closed tubular neighborhood of $\tau \subseteq \partial C$, then

$$\widetilde{C} = C \cup \bigcup_{\tau \subseteq \partial C} V(\tau).$$

Denote by $i : \widetilde{C} \hookrightarrow M_g^2$ the inclusion map.

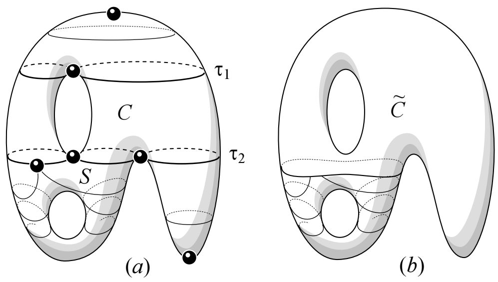

Figure 6: (a) $M_g^2 = C \cup S$; S is a torus with winding; $\tau_1 = \gamma$ is a compact singular leaf, τ_2 is a part of a non-compact singular leaf. (b) $\widetilde{C}$ is a small closed neighborhood of C, $\partial\widetilde{C} = S^1$.

4.1. Isotropic rank $h(C)$ of the compact part

Recall that $h(C) = h(H_1(C))$; see (11). Obviously, $\bigcup \partial\mathcal{C}_j^{max} \subset C$, moreover, it defines a maximal isotropic subgroup in C:

Lemma 22. $h(C) = h_c(\omega)$

Proof. Since $\bigcup \partial\mathcal{C}_i^{max} \subset C$, it holds $h_c(\omega) \leq h(C)$. Consider a maximal subgroup $G \subset i_* H_1(C)$ such that $H_\omega \subset i_* G$. By (15) we have $i_* G \subset G_c$, which implies $\mathrm{rk}(i_* G) \leq h_c(\omega)$. By Lemma 16 it holds $h(C) = \mathrm{rk}(i_* G)$, so $h(C) = h_c(\omega)$. □

Therefore it is sufficient to consider a maximal independent isotropic system of cycles in $\bigcup \partial\mathcal{C}_j^{max}$; their number is $h_c(\omega)$. These cycles are of three types:

- those induced from maximal components C_i^{max}, such as the side circles of τ_2 in Figure 6; their number is $c(\omega)$;
- those not induced from maximal components but induced from minimal components C_i^{min}, such as the middle circle of τ_2 in Figure 6; their number is denoted below by Δ;
- own cycles of $\bigcup \partial\mathcal{C}_j^{max}$ not induced from anywhere else, such as the cycles in Figure 5; the number of such cycles in a given $\tau \subseteq \bigcup \partial\mathcal{C}_j^{max}$ is $g(V(\tau))$, the genus of its small closed tubular neighborhood.

The following theorem formalizes these considerations and characterizes $c(\omega)$ and $h_c(\omega)$ from (13). Denote $\mathcal{D} = \langle i_*[\partial_j^\tau] \mid \tau \subseteq \bigcup \partial\mathcal{C}_j^{max} \rangle \subseteq H_1(M)$, where $\partial_j^\tau \subseteq \partial V(\tau)$ are connected components of $\partial V(\tau)$. Denote $\mathcal{D}_c = \mathcal{D} \cap G_c$ and $\Delta = \mathrm{rk}(\mathcal{D}/\mathcal{D}_c)$.

Note that

$$\mathrm{rk}\, \mathcal{D}_c = c(\omega). \tag{16}$$

Indeed, by definition $\mathrm{rk}\, \mathcal{D}_c \leq c(\omega)$. On the other hand, each maximal component contains a curve ∂_j^τ and, obviously, $i_*[\partial_j^\tau] \in \mathcal{D}_c$, so $c(\omega) \leq \mathrm{rk}\, \mathcal{D}_c$.

Proposition 23. *For a Morse form ω it holds*

$$h_c(\omega) = c(\omega) + \sum_{\tau \subseteq \bigcup \partial\mathcal{C}_j^{max}} g(V(\tau)) + \Delta. \tag{17}$$

Proof. Recall that $h_c(\omega) = h(G_c)$, where $G_c = i_* H_1\left(\bigcup \partial\mathcal{C}_j^{max}\right)$. Consider a small closed tubular neighborhood $V(\tau)$ of each $\tau \subseteq \bigcup \partial\mathcal{C}_j^{max}$ such that $V(\tau) \cap V(\tau') = \emptyset$. Note that $H_1(\tau) = H_1(V(\tau))$.

Let us construct a maximal isotropic subgroup $H_\tau \subseteq H_1(V(\tau))$ as in Proposition 17, i.e., $H_\tau = \langle [\alpha_i^\tau] \rangle \oplus \langle [\partial_j^\tau] \rangle$, where α_i^τ are those curves from Definition 6 that define the genus of $V(\tau)$ and ∂_j^τ are connected components of $\partial V(\tau)$. Consider

$$H = \langle i_* H_\tau \mid \tau \subseteq \bigcup \partial\mathcal{C}_j^{max} \rangle \subset G_c.$$

It is easy to see that H is a maximal isotropic subgroup, so $\operatorname{rk} H = h_c(\omega)$.

By construction,

$$H = \langle i_*[\alpha_i^\tau]\rangle \oplus \langle i_*[\partial_j^\tau]\rangle. \tag{18}$$

By Corollary 8, all $i_*[\alpha_i^\tau]$ are independent, thus

$$\operatorname{rk}\langle i_*[\alpha_i^\tau]\rangle = \sum_\tau g(V(\tau)). \tag{19}$$

Recall that $\langle i_*[\partial_j^\tau]\rangle = \mathcal{D}$, which is free abelian. By (16) we have $\operatorname{rk}\mathcal{D} = c(\omega) + \Delta$, where $\Delta = \operatorname{rk}(\mathcal{D}/\mathcal{D}_c)$, which together with (18) and (19) gives (17). □

Remark 24. For a weakly generic form,

$$h_c(\omega) = c(\omega) + \sum_\gamma g(V(\gamma)),$$

where γ are compact singular leaves.

Indeed, if ω is very weakly generic then in Proposition 23 each $\tau \subseteq \partial C$ is S^1 and attaches one maximal component, so $V(\tau)$ is a cylinder; thus $g(V(\tau)) = 0$ and $\Delta = 0$. If $\tau \not\subseteq \partial C$ then it is a compact singular leaf γ.

The following fact improves on Lemma 18:

Corollary 25. *If $c(\omega) = h_c(\omega)$ then for any compact singular leaf γ it holds $g(V(\gamma)) = 0$. For very weakly generic forms the converse is also true: if $g(V(\gamma)) = 0$ for all γ then $c(\omega) = h_c(\omega)$.*

4.2. The form's rank and the structure of the compact part

Since for any leaf γ it holds $\int_\gamma \omega = 0$, the compact part C of the foliation includes zero periods of ω and therefore defines its rank.

Proposition 26. *For a Morse form ω on M_g^2 it holds*

$$\operatorname{rk}\omega + c(\omega) \le 2\left(g - \sum_\tau g(V(\tau))\right). \tag{20}$$

In particular, if $\operatorname{rk}\omega \ge 2g-1$ then all $g(V(\tau)) = 0$.

Proof. For any $z \in G_c = i_* H_1\left(\bigcup \partial\mathcal{C}_j^{max}\right)$ it holds $\int_z \omega = 0$; so

$$\operatorname{rk}\omega \le 2g - \operatorname{rk} G_c. \tag{21}$$

Consider a small closed tubular neighborhood $V(\tau)$ of each $\tau \subseteq \bigcup \partial\mathcal{C}_j^{max}$, such that $V(\tau)\cap V(\tau') = \emptyset$. Then $H_1(\tau) = H_1(V(\tau))$, so for any $z \in i_* H_1(V(\tau))$ it holds $\int_z \omega = 0$.

Choose the curves $\alpha_j^\tau, \beta_j^\tau \subset V(\tau)$ as in Proposition 17, i.e., such that α_j^τ define $g(V(\tau))$ and $[\alpha_i^\tau]\cdot[\beta_j^\tau] = \delta_{ij}$. Let $G = \langle i_*[\alpha_j^\tau], i_*[\beta_j^\tau], \mathcal{D}_c\rangle \subseteq G_c$. Obviously, $\operatorname{rk} G \le \operatorname{rk} G_c$, so by Lemma 7, Proposition 17, and (16) we have $\operatorname{rk} G = 2\sum_\tau g(V(\tau)) + c(\omega)$, which together with (21) gives (20).

If $\operatorname{rk}\omega \ge 2g-1$ then $2\sum_\tau g(V(\tau)) \le 1 - c(\omega) \le 1$, so $\sum_\tau g(V(\tau)) = 0$. □

Corollary 27. *It holds $\sum g(V(\tau)) \le g$. If $\sum g(V(\tau)) = g$ then the form is exact: $\omega = df$, the foliation is compact, in particular, all τ are compact singular leaves: $\tau = \gamma$, and $c(\omega) = 0$.*

The first fact follows also from Corollary 8. Proposition 26 and Theorem 43 below imply:

Corollary 28. *For compactifiable $\mathcal{F}_\omega$ it holds $\operatorname{rk}\omega \le c(\omega)$, or, in terms of compact singular leaves:*

$$\operatorname{rk}\omega \le g - \sum g(V(\gamma)).$$

This Corollary improves on the result of Mel'nikova [16] who proved that $\operatorname{rk}\omega \le g$.

4.3. Calculation of $g(V(\gamma))$ in terms of singularities in γ

Denote by $d(\gamma)$ the number of maximal components glued to a compact singular leaf γ.

Lemma 29. $g(V(\gamma)) = 1 + \frac{1}{2}(|\gamma \cap \Omega_1| - d(\gamma))$.

Proof. The foliation $\mathcal{F}_\omega$ defines a foliation on $V(\gamma)$. Without loss of generality we can suppose that the connected components of $\partial V(\gamma)$ are leaves of $\mathcal{F}_\omega$. Glue up $\partial V(\gamma)$ by disks and continue the foliation to these disks with one center in each, so that the number of centers is $\deg\gamma$; see Figure 7. The constructed surface has a foliation with $d(\gamma)$ centers and $|\gamma\cap\Omega_1|$ conic singularities, while by (8) we have $|\gamma\cap\Omega_1| - d(\gamma) = 2g(V(\gamma)) - 2$. □

Corollary 30. $g(V(\gamma)) \geq 1$ *iff* $|\gamma \cap \Omega_1| \geq d(\gamma)$.

Remark 31. If ω is weakly generic then for any γ it holds $|\gamma \cap \Omega_1| = 1$ and $d(\gamma) = 3$, so $g(V(\gamma)) = 0$.

4.4. Compact singular leaf as a graph

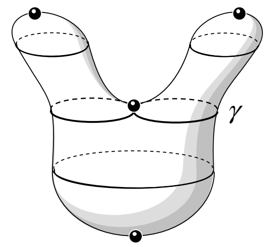

Figure 7: A tubular neighborhood $V(\gamma)$ of a compact singular leaf γ can be glued up by $\deg\gamma$ disks.

We can consider a compact singular leaf γ as a graph; loops and multiple edges are allowed. Recall that

$$\gamma = \bigcup_{i=1}^{q} \gamma_i^0 \cup \bigcup_{j=1}^{p} s_j,$$

where γ_i^0 are compactifiable leaves of $\mathcal{F}_\omega$, and $s_j \in \operatorname{Sing}\omega$ are singularities. Vertices of the graph are singularities s_j and edges are compactifiable leaves γ_i^0. This gives information concerning the structure of compact singular leaves:

Lemma 32. *A compact singular leaf γ contains an even number of compactifiable leaves, $q = 2|\gamma \cap \Omega_1|$. In addition,* $\operatorname{rk} H_1(\gamma) = 1 + |\gamma \cap \Omega_1|$.

Proof. For the number of vertices p it holds $p = |\gamma\cap\Omega_1|$. Since all singularities are conic, we have $\deg s_j = 4$ for all s_j, i.e γ is a 4-regular graph. So by Euler theorem, the number of edges (in our case, compactifiable leaves) is even: $q = 2|\gamma \cap \Omega_1|$. For the cycle rank $m(\gamma)$ we have $m(\gamma) = q - p + 1 = |\gamma \cap \Omega_1| + 1$ [8]; on the other hand, $m(\gamma) = \operatorname{rk} H_1(\gamma)$. □

For example, the complete graph on five vertices K_5 can be a compact singular leaf; the number of vertices $p(K_5) = 5$, its cycle rank $m(K_5) = 6$; see Figure 8.

Definition 33. The *genus of a graph* γ is the minimum genus of a surface M_g^2 in which the graph can be embedded without any crossings. A *planar graph* has genus 0.

Recall that $V(\gamma)$ is a small tubular neighborhood of γ; obviously, $g(\gamma) \leq g(V(\gamma))$. Consider all compact singular leaves γ as graphs. Then

$$\sum_\gamma g(\gamma) \leq \sum_\gamma g(V(\gamma)) \leq g. \tag{22}$$

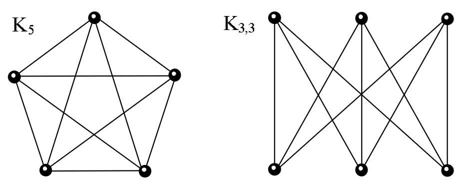

Figure 8: A graph is planar iff it contains no subgraph homeomorphic to one of these two graphs.

Figure 5 gives an example of a strict inequality: a planar singular leaf with $g(V(\gamma)) = 1$. Obviously, $g(V(\gamma)) = 0$ implies that γ is planar.

For example, $g(K_5) = 1$.

Kuratowski's theorem states that a finite graph is planar if and only if it does not contain any subgraph homeomorphic (equal up to vertices of degree two) to K_5 or $K_{3,3}$ [8]; see Figure 8. In particular, if $|\gamma\cap\Omega_1| \leq 4$ then the graph is planar. For example, if ω is generic or weakly generic then each compact singular leaf γ is a figure of eight, which is planar. While the number of planar singular leaves is unlimited, there can be only few non-planar ones:

Lemma 34. *Let n be the number of non-planar compact singular leaves. Then:*
(i) $n \le g$;
(ii) $n \le \frac{1}{5}|\Omega_1| = \frac{1}{5}(2g - 2 + |\Omega_0|)$;
(iii) *If the total number of compact singular leaves* $|\{\gamma\}| \ge |\Omega_1| - 3$, *then* $n = 0$.

PROOF. (i), and even stronger $n \le \sum_\gamma g(V(\gamma))$, follows from (22). (ii) follows from Kuratowski's theorem: a non-planar graph has at least 5 vertices; the equality in (ii) is by (8).

(iii): Suppose there exists a non-planar leaf γ; then $|\gamma \cap \Omega_1| \ge 5$, so $\sum_\gamma |\gamma \cap \Omega_1| \ge 5 + |\{\gamma\}| - 1$. Since $|\Omega_1| \ge \sum_\gamma |\gamma \cap \Omega_1|$, we obtain $|\{\gamma\}| \le |\Omega_1| - 4$, a contradiction. □

The topology of one compact singular leaf influences the topology of the whole foliation. Indeed, Proposition 23 and Proposition 26 imply:

Corollary 35. *If there exists non-planar a compact singular leaf, i.e.,* $g(\gamma) \ge 1$, *then*
(i) $c(\omega) < h_c(\omega)$,
(ii) $c(\omega) \le 2g - 2 - \operatorname{rk}\omega$,
(iii) $\operatorname{rk}\omega \le 2g - 2$.

For example, a foliation with a non-planar compact singular leaf on a torus T^2 is compactifiable.

5. Topology of the minimal part of the foliation

Recall that $M_g^2 = C \cup S$, where $C = \bigcup \overline{\mathcal{C}_j^{max}}$ is the compact part of the foliation and $S = \bigcup \overline{\mathcal{C}_i^{min}}$ is its minimal part; $C \cap S = \partial C = \partial S$. In this section we study the minimal part S: namely, we construct its maximal isotropic subgroup and calculate its genus $g(S)$.

5.1. Maximal isotropic subgroup of the minimal part

The boundary of S may have singularities, so we consider its small closed tubular neighborhood $\widetilde{S}$ such that $\partial\widetilde{S}$ is non-singular and consists of non-intersecting circles. It has the same homology group $H_1(\widetilde{S}) = H_1(S)$ and genus $g(\widetilde{S}) = g(S)$.

Namely, $\widetilde{S}$ is constructed as follows: For each connected component $\tau \subseteq \partial S = \partial C$ of the boundary, consider its small closed neighborhood $V(\tau)$. Then

$$\widetilde{S} = S \cup \bigcup_{\tau \subseteq \partial S} V(\tau).$$

Associate with each $V(\tau)$ a maximal isotropic subgroup $H_\tau \subseteq H_1(V(\tau))$. By Proposition 17, we can choose $H_\tau = \langle[\alpha_i^\tau]\rangle \oplus \langle[\partial_j^\tau]\rangle$, where the curves α_i^τ from Definition 6 define the genus of $V(\tau)$ and ∂_j^τ are connected components of $\partial V(\tau)$.

Consider the curves $\alpha_i^\tau \subset V(\tau)$ in connection with $\widetilde{S}$:

Lemma 36. *The set $\widetilde{S} \setminus \bigcup_{\tau \subseteq \partial S, i} \alpha_i^\tau$ has the same number of connected components as $\widetilde{S}$. In addition, a system $\{\alpha_i^\tau\}$ defining the genus $g(\bigcup_{\tau \subseteq \partial S} V(\tau))$ can be extended to a system of curves $\{\alpha_i^S\}$ defining the genus $g(S)$, i.e. $\{\alpha_i^\tau\} \subset \{\alpha_i^S\}$.*

PROOF. Without loss of generality suppose that $\widetilde{S}$ is connected (otherwise consider one connected component). By construction,

$$\widetilde{S} \setminus \bigcup_{\tau \subseteq \partial S, i} \alpha_i^\tau = \left(S \setminus \bigcup_{\tau, i} \alpha_i^\tau \right) \cup \bigcup_{\tau \subseteq \partial S} \left(V(\tau) \setminus \bigcup_i \alpha_i^\tau \right).$$

Denote $S^0 = S \setminus \bigcup_{\tau,i} \alpha_i^\tau$ and $V_\tau^0 = V(\tau) \setminus \bigcup_i \alpha_i^\tau$. Since $\alpha_i^\tau \subseteq \partial S$, the set S^0 is connected; by definition of α_i^τ, the sets V_τ^0 are also connected. In addition, $S^0 \cap V_\tau^0 \ne \emptyset$ for all $\tau \subseteq \partial S$. This implies that $S^0 \cup V_\tau^0$ is connected, and so is $S^0 \cup \bigcup_\tau V_\tau^0$. Therefore $\widetilde{S} \setminus \bigcup_{\tau,i} \alpha_i^\tau$ is connected as well. □

Now consider the curves $\partial_j^\tau \subseteq \partial V(\tau)$ in connection with $\widetilde{S}$. Denote $j : \widetilde{S} \hookrightarrow M_g^2$.

Lemma 37. *For any system of curves $\{\alpha_i^S\}$ defining the genus $g(S)$ it holds*

$$\langle j_*[\partial_i^\tau] \mid \tau \subseteq \partial S\rangle \subseteq \langle j_*[\alpha_i^S]\rangle \oplus H_\omega.$$

Proof. Since $\tau \subseteq \partial S$, there exist two types of boundary components $\partial_i^\tau \subseteq \partial V(\tau)$: $\partial_i^\tau(S) \subset \operatorname{int}(S)$ and $\partial_i^\tau(C) \subset \operatorname{int}(C)$. Obviously, $\partial_i^\tau(C) \sim \gamma_i$ for some compact non-singular leaf γ_i.

Let $\widetilde{S} = \bigcup_{i=1}^k \widetilde{S}_i$, where $\widetilde{S}_i$ are its connected components. Denote by $\widetilde{S}'$ the result of cutting $\widetilde{S}$ open along the curves α_j^S; then $\widetilde{S}'$ has the same number of connected components: $\widetilde{S}' = \bigcup_{i=1}^k \widetilde{S}'_i$. Note that all $\widetilde{S}'_i$ are spheres with holes and $\partial\widetilde{S}'_i \subseteq \partial\widetilde{S} \cup \bigcup \alpha_j^{S\pm}$. In addition, $\partial\widetilde{S} = \bigcup \partial_i^\tau(C) = \bigcup \gamma_i$, where $\gamma_i \subset C$ are compact non-singular leaves.

Each $\partial_i^\tau = \partial_i^\tau(S) \subset \operatorname{int}(S)$ splits up some $\widetilde{S'_j}$, so we obtain $[\partial_i^\tau(S)] \in \langle[\alpha_i^S], [\partial_j^\tau(C)]\rangle$, i.e. $j_*[\partial_i^\tau] \in \langle j_*[\alpha_i^S], H_\omega\rangle$. Let $z \in \langle j_*[\alpha_i^S]\rangle \cap H_\omega$, then

$$z = \sum m_i j_*[\alpha_i^S] = \sum l_i[\gamma_i].$$

Consider $\beta_k^S \subset \widetilde{S}$ such that $[\alpha_i^S]\cdot[\beta_k^S] = \delta_{ik}$ (Proposition 17); then $z \cdot j_*[\beta_k^S] = m_k = 0$, since $\gamma_i \cap \beta_k^S = \emptyset$. This gives $z = 0$ and thus $\langle j_*[\alpha_i^S], H_\omega\rangle = \langle j_*[\alpha_i^S]\rangle \oplus H_\omega$. □

Now let us construct a maximal isotropic subgroup $H_{\widetilde{S}} \subseteq H_1(\widetilde{S}) = H_1(S)$. Suppose that the curves α_i^τ define the genus $g(\bigcup_{\tau\subseteq\partial S} V(\tau))$. By Lemma 36, we can choose the system $\{\alpha_i^S\}$ such that $\{\alpha_i^S\} \supset \{\alpha_i^\tau\}$. Denote $j_\tau : V(\tau) \hookrightarrow \widetilde{S}$.

Proposition 38. *The subgroup*

$$H_{\widetilde{S}} = \big\langle [\alpha_i^S], j_{\tau*}H_\tau \mid i = 1, \dots, g(S),\, \tau \subseteq \partial S\big\rangle \subset H_1(\widetilde{S})$$

is maximal isotropic.

Proof. By construction, all curves $\alpha_i^\tau \subset V(\tau)$, $\tau \subseteq \partial S$, are included in the system $\{\alpha_i^S\}$ defining the genus. The remaining curves α_i^S can be chosen such that $\alpha_i^S \cap V(\tau) = \emptyset$, so the subgroup $H_{\widetilde{S}}$ is isotropic. Consider a cycle $z \in H_1(\widetilde{S})$ such that $z \cdot H_{\widetilde{S}} = 0$ and realize it by a curve $\alpha \subset \widetilde{S}$ with $\alpha \cap \alpha_i^S = \alpha \cap \partial_i^\tau = \emptyset$.

Denote by $\widetilde{S}'$ the result of cutting $\widetilde{S}$ open along the curves $\alpha_i^S \subset S \setminus \bigcup_\tau V(\tau)$ and ∂_i^τ; it breaks up into some neighborhoods $V(\tau_j)$ and some S'_j which are spheres with holes, $\widetilde{S}' = \bigcup V(\tau_j) \cup \bigcup S'_i$. Let $\alpha_0 \subseteq \alpha$ be a connected component. If $\alpha_0 \subset V(\tau_j)$, the condition $[\alpha_0]\cdot H_{\tau_j} = 0$ implies $[\alpha_0] \in j_{\tau_j *}H_{\tau_j} \subset H_{\widetilde{S}}$, since the subgroup H_{τ_j} is maximal. If $\alpha_0 \subset S'_j$, it splits S'_j up and thus is induced from $\partial S'_j \subseteq \bigcup(\alpha_i^{S+} \cup \alpha_i^{S-}) \cup \bigcup_{\tau,i} \partial_i^\tau$, where $\alpha_i^{S\pm}$ are two copies of α_i^S. This implies $[\alpha_0] \in \langle[\alpha_i^S], j_{\tau*}[\partial_j^\tau]\rangle \subseteq H_{\widetilde{S}}$.

Recall that $\alpha = \bigcup \alpha_0$ is the union of its connected components, so $z = [\alpha] \in H_{\widetilde{S}}$ and therefore the subgroup $H_{\widetilde{S}}$ is maximal. □

5.2. Genus of the minimal part

Recall that $M_g^2 = C \cup S$. We have constructed maximal isotropic subgroups of rank $h_c(\omega)$ in the compact part C and of rank $g(S)$ in the minimal part $\operatorname{int}(S)$, respectively. They combine into a maximal isotropic subgroup $H \subset H_1(M_g^2)$ of rank g. However, they have some cycles in common:

Proposition 39. *For a Morse form ω on M_g^2,*

$$h_c(\omega) + g(S) - \sum_{\tau\subseteq\partial S=\partial C} g(V(\tau)) - \Delta = g. \tag{23}$$

Proof. Consider $\widetilde{C} = C \cup \bigcup_{\tau\subseteq\partial C} V(\tau)$ and $\widetilde{S} = S \cup \bigcup_{\tau\subseteq\partial S} V(\tau)$. In Propositions 23 and 38 we have constructed maximal isotropic subgroups $H_{\widetilde{C}} \subset H_1(\widetilde{C})$ and $H_{\widetilde{S}} \subset H_1(\widetilde{S})$, respectively. Let $i : \widetilde{C} \hookrightarrow M_g^2$ and $j : \widetilde{S} \hookrightarrow M_g^2$ be inclusion maps. Then $i_*H_{\widetilde{C}} \subseteq i_*H_1(\widetilde{C})$ and $j_*H_{\widetilde{S}} \subset j_*H_1(\widetilde{S})$ are also maximal isotropic subgroups. By Proposition 23 we have

$$i_*H_{\widetilde{C}} = \langle i_*[\alpha_i^\tau], i_*[\partial_i^\tau] \mid \tau \subseteq \bigcup \partial\mathcal{C}_j^{max}\rangle, \tag{24}$$

where α_i^τ define the genus of $V(\tau)$ and ∂_i^τ form its boundary. By Proposition 38,

$$j_*H_{\widetilde{S}} = \langle j_*[\partial_i^\tau], j_*[\alpha_i^S] \mid \tau \subseteq \partial S\rangle,$$

where α_i^S define the genus of S. In addition, by Lemma 36 we can choose the system $\{\alpha_i^S\}$ such that $\{\alpha_i^S\} \supset \{\alpha_i^\tau\}$, where $\tau \subseteq \partial S$. Since $i(\widetilde{C} \cap \widetilde{S}) = j(\widetilde{C} \cap \widetilde{S})$, we have $i_*[\partial_i^\tau] = j_*[\partial_i^\tau]$ for any $\tau \subseteq \partial S$. Denote

$$H = \langle i_*H_{\widetilde{C}}, j_*H_{\widetilde{S}}\rangle = \langle i_*[\alpha_i^\tau], i_*[\partial_i^\tau], j_*[\alpha_i^S] \mid \tau \subseteq \bigcup \partial\mathcal{C}_j^{max}\rangle. \tag{25}$$

Obviously, $H \subset H_1(M_g^2)$ is isotropic; let us show that it is maximal.

Consider $z \in H_1(M_g^2)$, $z \cdot H = 0$; in particular, $z \cdot i_*[\partial_i^\tau] = 0$, $\tau \subseteq \partial S$, so $z = [\alpha_C] + [\alpha_S] + [\alpha_V]$, where $\alpha_C \subset \operatorname{int}(C)$, $\alpha_S \subset \operatorname{int}(S)$, and $\alpha_V \subset \bigcup_{\tau\subseteq\partial S} \operatorname{int}(V(\tau))$. Without loss of generality we can assume that $\alpha_C \cap \widetilde{S} = \emptyset$ and $\alpha_S \cap \widetilde{C} = \emptyset$. The equation $z \cdot H = 0$ implies $z \cdot i_*H_{\widetilde{C}} = 0$, so $([\alpha_C] + [\alpha_V]) \cdot i_*H_{\widetilde{C}} = 0$. Since $\alpha_C \cup \alpha_V \subset \widetilde{C}$, the cycle $[\alpha_C]+[\alpha_V] \in i_*H_1(\widetilde{C})$. The subgroup $i_*H_{\widetilde{C}}$ is maximal, thus $[\alpha_C]+[\alpha_V] \in i_*H_{\widetilde{C}} \subseteq H$.

On the other hand, $z \cdot j_*[\alpha_i^S] = 0$. Recall that by construction there are two kinds of α_i^S: the curves $\alpha_i^S \subset \operatorname{int}(S)$ and $\alpha_i^S \subset \operatorname{int}(V(\tau))$, $\tau \subseteq \partial S$. The former set implies that $[\alpha_S] \cdot j_*[\alpha_i^S] = 0$. Since also $[\alpha_S] \cdot i_*[\partial_i^\tau] = 0$, we obtain $[\alpha_S] \in j_*H_{\widetilde{S}} \subseteq H$. Therefore $z = [\alpha_C] + [\alpha_S] + [\alpha_V] \in H$, i.e., $H \subset H_1(M_g^2)$ is maximal; by (11), $\operatorname{rk} H = g$.

By (24) and (25) we have

$$\operatorname{rk} H = \operatorname{rk}(i_*H_{\widetilde{C}}) + \operatorname{rk}\langle j_*[\alpha_i^S]\rangle - \operatorname{rk}(i_*H_{\widetilde{C}} \cap \langle j_*[\alpha_i^S]\rangle) = g. \tag{26}$$

Lemma 22 gives $\operatorname{rk}(i_*H_{\widetilde{C}}) = h_c(\omega)$ and Corollary 8 gives $\operatorname{rk}\langle j_*[\alpha_i^S]\rangle = g(S)$. So

$$h_c(\omega) + g(S) - \operatorname{rk}(i_*H_{\widetilde{C}} \cap \langle j_*[\alpha_i^S]\rangle) = g. \tag{27}$$

Now we only need to calculate $\operatorname{rk}(i_*H_{\widetilde{C}} \cap \langle j_*[\alpha_i^S]\rangle)$. By (24),

$$i_*H_{\widetilde{C}} \cap \langle j_*[\alpha_i^S]\rangle = \left(\langle i_*[\alpha_i^\tau]\rangle \cap \langle j_*[\alpha_i^S]\rangle\right) \cup \left(\langle i_*[\partial_i^\tau]\rangle \cap \langle j_*[\alpha_i^S]\rangle\right).$$

The union is disjoint since $\alpha_i^\tau \cap\ \partial_j^{\tau'} = \emptyset$, thus

$$\operatorname{rk}(i_*H_{\widetilde{C}} \cap \langle j_*[\alpha_i^S]\rangle) = \operatorname{rk}(\langle i_*[\alpha_i^\tau]\rangle \cap \langle j_*[\alpha_i^S]\rangle) + \operatorname{rk}(\langle i_*[\partial_i^\tau]\rangle \cap \langle j_*[\alpha_i^S]\rangle). \tag{28}$$

By construction, for the first summand we have

$$\operatorname{rk}(\langle i_*[\alpha_i^\tau]\rangle \cap \langle j_*[\alpha_i^S]\rangle) = \sum_{\tau\subset\partial S} g(V(\tau)). \tag{29}$$

Consider the second summand in (28). Recall that $\mathcal{D} = \langle i_*[\partial_i^\tau] \mid \tau \subseteq \bigcup \partial\mathcal{C}_j^{max}\rangle$ and $\mathcal{D}_c \subseteq \mathcal{D}$ is a subgroup of elements homologous to a union of compact leaves; cf. Proposition 23. Thus $\langle i_*[\partial_i^\tau]\rangle \cap \langle j_*[\alpha_i^S]\rangle = \mathcal{D} \cap \langle j_*[\alpha_i^S]\rangle$.

Let $z_1, \dots, z_k \in \mathcal{D} \cap \langle j_*[\alpha_i^S]\rangle$ be a basis. Consider $z_i + \mathcal{D}_c \in \mathcal{D}/\mathcal{D}_c$ and suppose that $\sum n_i(z_i + \mathcal{D}_c) = 0$, i.e., $z = \sum n_i z_i \in \mathcal{D}_c$. Then there exist compact non-singular leaves $\gamma_i \subset C \setminus \widetilde{S}$ such that $z = \sum l_i[\gamma_i]$. On the other hand, $z = \sum m_i j_*[\alpha_i^S]$. We obtain

$$z = \sum l_i[\gamma_i] = \sum m_i j_*[\alpha_i^S].$$

Consider closed curves $\beta_k^S \subset \widetilde{S}$ such that $[\alpha_i^S] \cdot [\beta_k^S] = \delta_{ik}$ (Proposition 17); then $z \cdot j_*[\beta_k^S] = m_k = 0$ since $\gamma_i \cap \beta_k^S = \emptyset$. This gives $z = 0$ and all $n_i = 0$, i.e., $z_i + \mathcal{D}_c$ are independent. Thus $\operatorname{rk}(\mathcal{D} \cap \langle j_*[\alpha_i^S]\rangle) \leq \operatorname{rk}(\mathcal{D}/\mathcal{D}_c) = \Delta$.

Now let $u_1 + \mathcal{D}_c, \dots, u_\Delta + \mathcal{D}_c$ be a basis of $\mathcal{D}/\mathcal{D}_c$. Then $u_i \notin \mathcal{D}_c$ and we can choose $u_i \in \langle j_*[\partial_i^\tau] \mid \tau \subseteq \partial S\rangle \subseteq \langle j_*[\alpha_i^S]\rangle \oplus H_\omega$ (Lemma 37). Since $u_i \notin \mathcal{D}_c$, we have $u_i \in \langle j_*[\partial_i^\tau] \mid \tau \subseteq \partial S\rangle \cap \langle j_*[\alpha_i^S]\rangle \subseteq \mathcal{D} \cap \langle j_*[\alpha_i^S]\rangle$, i.e. $\Delta \leq \operatorname{rk}(\mathcal{D} \cap \langle j_*[\alpha_i^S]\rangle)$.

Thus $\operatorname{rk}(\mathcal{D} \cap \langle j_*[\alpha_i^S]\rangle) = \Delta$, which together with (27–29) gives (23). □

Remark 40. For a very weakly generic form,

$$h_c(\omega) + g(S) = g.$$

The proof is as in Remark 24.

Finally, together with (13), Proposition 39 gives:

Corollary 41. *If $\mathcal{F}_\omega$ is compactifiable then*

$$h_c(\omega) = h(\ker[\omega]) = g.$$

6. Structure theorem

Our main result summarizes our study of geometry of minimal components and compact singular leaves of a Morse form foliation:

Theorem 42. *Let ω be a Morse form on M_g^2. Then*

$$c(\omega) + \sum_\gamma g(V(\gamma)) + g\left(\bigcup_{i=1}^{m(\omega)} \overline{C_i^{min}}\right) = g, \tag{30}$$

where $c(\omega)$ is the number of homologically independent compact non-singular leaves; $m(\omega)$ is the number of minimal components; $V(*)$ is a small closed tubular neighborhood; and $g(*)$ is the genus of a surface. The summation is taken over all compact singular leaves γ.

The proof is obtained by summing up the results of Propositions 23 and 39.

This theorem immediately gives an important criterion for compactifiability of an arbitrary Morse form foliation; cf. [15]:

Theorem 43. *The foliation is compactifiable iff*

$$c(\omega) + \sum_\gamma g(V(\gamma)) = g.$$

Together with Lemma 9, Theorem 42 gives an upper bound on the number of minimal components $m(\omega)$ that is better than one given by (1):

Corollary 44. *For a Morse form ω it holds*

$$c(\omega) + m(\omega) \leq g - \sum_\gamma g(V(\gamma)).$$

Lemma 9 and Remark 31 refine Theorem 42 for (very) weakly generic forms, respectively:

Corollary 45. *For a very weakly generic form,*

$$c(\omega)+\sum_{\gamma} g(V(\gamma))+\sum_{i=1}^{m(\omega)} g(\overline{C_i^{min}})=g.$$

Corollary 46. *For a weakly generic form,*

$$c(\omega)+\sum_{j=1}^{m(\omega)} g(\overline{\mathcal{C}_j^{min}})=g.$$

7. Minimal components of a very weakly generic form in terms of singularities

We will study the relation of topology of minimal components of very weakly generic forms with the number of singularities inside them. Namely, we calculate the genus $g(\overline{\mathcal{C}_i^{min}})$ and the number of minimal components $m(\omega)$ for such forms in terms of $k(\omega)$—the number of singularities inside $\mathcal{C}_i^{min}$.

7.1. Singularities of a very weakly generic form inside minimal components

Recall that we say that a singularity s is *inside* a component $\mathcal{C}$ if $s\in \operatorname{int}(\overline{\mathcal{C}})$.

Theorem 47. *Let ω be a very weakly generic form on M_g^2, $\mathcal{C}^{min}$ a minimal component, and $k=|\operatorname{Sing}\omega\cap \operatorname{int}(\overline{\mathcal{C}^{min}})|$. Then*

$$g(\overline{\mathcal{C}^{min}})=1+\frac{k}{2}.$$

In particular, the number of singularities k inside a minimal component is even.

Proof. Denote $\mathcal{C}=\mathcal{C}^{min}$. Since each connected component ∂_j of $\partial\overline{\mathcal{C}}$ contains one singularity, by Lemma 1 it locally attaches one maximal component $\mathcal{C}_j^{max}$ to $\mathcal{C}$. Cut each $\mathcal{C}_j^{max}$ off $\mathcal{C}$, replacing it with a disk D_j^2, and continue the foliation to these disks with one center in each. Let this new manifold M' have genus g'; obviously, $g'=g(\overline{\mathcal{C}})$. Apart from k conic singularities inside $\mathcal{C}$, the foliation on M' has a center in each D_j^2 and a conic singularity on ∂D_j^2. By the Poincaré—Hopf theorem, $k=2g'-2$. □

The condition for the form to be very weakly generic is important: for a not very weakly generic form the number of singularities k inside a minimal component does not have to be even. A counter-example is shown in Figure 9: a unique singularity q inside the only minimal component on a double torus.

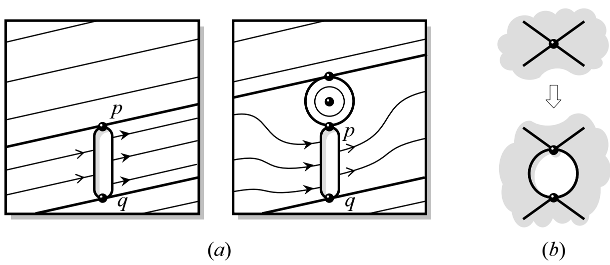

Figure 9: (a) Not very weakly generic foliation on $T^2 \sharp T^2$, obtained by transforming the generic foliation shown in Figure 3 by adding a center as shown in (b); shaded is the minimal component.

Lemma 9 gives:

Corollary 48. *For a very weakly generic form,*

$$g\left(\bigcup_{i=1}^{m(\omega)} \overline{C_i^{min}}\right)=m(\omega)+\frac{k(\omega)}{2},$$

where $m(\omega)$ is the number of minimal components and $k(\omega)=|\operatorname{Sing}\omega\cap\bigcup\operatorname{int}(\overline{\mathcal{C}^{min}})|$ is the total number of singularities inside minimal components. In particular, $k(\omega)$ is even.

Let us study the properties of $k(\omega)$:

Proposition 49. *Let ω be a very weakly generic form on $M = M_g^2$, $g \geq 1$. Then:*

(i) *It holds*

$$0 \leq k(\omega) \leq 2g - 2; \tag{31}$$

on a given M all even values within these bounds are reached; in particular, the bounds are exact.

(ii) *$k(\omega) = 0$ (lowest) iff all minimal components are tori with holes (have genus one).*

(iii) *If $g \geq 2$, then $k(\omega) = 2g - 2$ (highest) iff there exists a minimal component with $g(\overline{\mathcal{C}^{min}}) = g$.*

The latter condition, obviously, means that there is exactly one minimal component $\mathcal{C}^{min}$ ($m(\omega) = 1$) with $\overline{\mathcal{C}^{min}}$ covering M with possible holes; in particular, $c(\omega) = 0$.

Proof. (i) Corollary 48 gives $k(\omega) \leq 2g$, while $k(\omega) = 2g$ implies $m(\omega) = 0$ and thus $k(\omega) = g = 0$. Since $k(\omega)$ is even, we have $k(\omega) \leq 2g - 2$. That all even values within the bounds are reached follows from Theorem 55 below.

(ii) This follows from Theorem 47.

(iii) If $g \geq 2$ then $k(\omega) = 2g - 2 \neq 0$, thus $m(\omega) \neq 0$, while Corollary 48 implies $m(\omega) \leq 1$. By Theorem 47, $g(\overline{\mathcal{C}^{min}}) = g$, i.e., $g(\overline{M \setminus \overline{\mathcal{C}^{min}}}) = 0$. The converse follows from Theorem 47. □

Note that the lower bound is reached, in particular, on compactifiable foliations, and the upper bound on minimal foliations.

7.2. Number of minimal components

There are known upper bounds on the number of minimal components: $m(\omega) \leq g$ [14], $m(\omega) \leq g - c(\omega)$ [6]. For a very weakly generic form, Corollaries 45 and 48 give an equality:

Theorem 50. *For a very weakly generic form,*

$$m(\omega) = g - \left(c(\omega) + \sum_{\gamma} g(V(\gamma)) + \frac{k(\omega)}{2} \right), \tag{32}$$

where $V(\gamma)$ is a small tubular neighborhood of a compact singular leaf γ.

It is known that for a weakly generic form it holds [7]:

$$c(\omega) + m(\omega) = g - \frac{k(\omega)}{2}. \tag{33}$$

The following corollary generalizes this fact and clarifies its geometrical meaning:

Corollary 51. *For a very weakly generic form,* (33) *holds iff $g(V(\gamma)) = 0$ for any compact singular leaf γ.*

Let the foliation have at most one minimal component and no singularities inside it:

Lemma 52. *For a very weakly generic form with $m(\omega) \leq 1$ and $k(\omega) = 0$,*

$$h(\ker[\omega]) = g - m(\omega). \tag{34}$$

Proof. Theorem 50 and Remark 24 imply $h_c(\omega) = g - m(\omega)$. If $m(\omega) = 0$, Corollary 21 gives $h(\ker[\omega]) = h_c(\omega)$. If $m(\omega) = 1$, by Proposition 49 (ii) the minimal component $\mathcal{C}^{min}$ is a torus with holes; thus $\operatorname{rk} \omega|_{\overline{\mathcal{C}^{min}}} = 2 = 2g(\overline{\mathcal{C}^{min}})$. Lemmas 19 and 20 imply $h(\ker[\omega]) = h_c(\omega)$. □

It can be difficult to calculate $g(V(\gamma))$ for compact singular leaves γ. We can, however, give lower and upper bounds on the number of minimal components $m(\omega)$ that are weaker than (32) but do not involve $g(V(\gamma))$:

Theorem 53. *For very weakly generic forms ω on M_g^2 it holds*

$$g - \frac{k(\omega)}{2} - h(\ker[\omega]) \leq m(\omega) \leq g - \frac{k(\omega)}{2} - c(\omega). \tag{35}$$

In addition,

(i) *if $k(\omega) > 0$, then $m(\omega) > 0$;*
(ii) *if $k(\omega) = 0$ and $h(\ker[\omega]) = g$, then $m(\omega) \neq 1$.*

On a given M the bounds given by the system (35) *and (i) are exact, and all intermediate values are reached except for the case specified in (ii).*

Note that if $k(\omega) = 0$ and $h(\ker[\omega]) = g$, then $m(\omega) = 0, \cancel{1}, 2, 3, \dots, g$. In addition, if $k(\omega) = 0$ then the left side of (35) is non-negative and the bound given by (35) alone is exact.

However, if $k(\omega) > 0$ then the left side of (35) can be negative. For example, if $\operatorname{rk}\omega = 2$ and the foliation is minimal, then $h(\ker[\omega]) \geq g-1$ (Corollary 15) and $k(\omega) = 2g-2$ (Theorem 47). So $g - \frac{k(\omega)}{2} - h(\ker[\omega]) \leq 2 - g < 0$ for $g \geq 3$. In this case, (i) gives a better bound.

Proof of Theorem 53. Theorem 50 and Remark 24 imply $m(\omega) = g - \frac{k(\omega)}{2} - h_c(\omega)$. Since (13) implies $c(\omega) \leq h_c(\omega) \leq h(\ker[\omega])$, we obtain (35). Item (i) is obvious; (ii) is by Lemma 52. Exactness of the bounds and existence of all intermediate values are shown in Theorem 55 below. In particular, Lemma 19 gives a sufficient condition for reaching the lower bound, and Corollary 51, upper. □

The value of $m(\omega)$ can vary by $h(\ker[\omega]) - c(\omega)$, cf. (13). We can also bound $m(\omega)$ in terms of $\operatorname{rk}\omega$:

Corollary 54. *For very weakly generic forms on M_g^2 it holds*

$$m(\omega) \geq \operatorname{rk}\omega - g - \frac{k(\omega)}{2};$$

if $\operatorname{rk}\omega = 2g$ (the highest), then

$$m(\omega) = g - \frac{k(\omega)}{2}.$$

Proof. Since $h(\ker[\omega]) \leq \operatorname{rk}\ker[\omega] = 2g - \operatorname{rk}\omega$, Theorem 53 gives the inequality, which by the upper bound in (35) turns into equality if $\operatorname{rk}\omega = 2g$. □

Though this bound is weaker than (35), it is easier to calculate. This bound is efficient for forms with large $\operatorname{rk}\omega$, which are the "majority" of all forms (in terms of measure).

8. Completeness of our results

Finally, we will show that we have completely characterized the relationships between the foliation characteristics $c(\omega)$ and $m(\omega)$ and the form's characteristics $k(\omega)$ and $h(\ker[\omega])$; in other words, that our system of relations between their values is complete and no new relations can be obtained without involving other variables.

Namely, we show that any combination of these values allowed by our results is realized even in the class of very weakly generic forms. In particular, our bounds are exact and all intermediate values are reached.

Theorem 55. *For any non-negative g, k, h, m, c that satisfy the relations* (31), (14), (34), *and* (35), *correspondingly, on M_g^2 there exists a very weakly generic form ω such that $k(\omega) = k$, $h(\ker[\omega]) = h$, $c(\omega) = c$, and $m(\omega) = m$. If c satisfies* (33), *then ω can be chosen generic.*

Proof. Recall what each relation states:

$$
\begin{array}{lrl}
 & 0 \leq & g, \\
(31): & 0 \leq k \leq & 2g-2 \qquad (k=0 \text{ if } g=0);\ k \text{ is even}, \\
(14): & 0 \leq h \leq & g, \\
(34): & c \leq h = & g-m \qquad \text{if } k=0 \text{ and } m \leq 1, \\
(35): & g-\frac{1}{2}k-h \leq m \leq & g-\frac{1}{2}k-c \quad (m \neq 0 \text{ if } k \neq 0), \\
(33): & m = & g-\frac{1}{2}k-c.
\end{array}
$$

If $g=0$ then $k=m=0$ and the statement trivially holds, so we assume $g>0$. In the rest of the proof we assume that all unspecified periods of ω are incommensurable.

Case $k=0$:

Figure 10 shows a connected sum $\sharp$ of m tori $T_i^{(m)}$ with minimal foliation—cf. Proposition 49 (ii)—and $h_c = g-m$ tori $T_j^{(c)}$ with a compactifiable foliation, among them c ones with a compact foliation, $c(\omega_j)=1$, and the rest as in Figure 5, $c(\omega_j)=0$. Denote the constructed manifold with such foliation by $M(c,h_c,m)$.

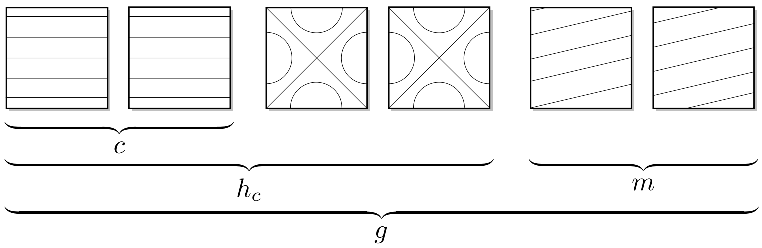

Figure 10: Construction of $M(c,h_c,m)$ in Theorem 55 for the case $k=0$.

Consider characteristics of this foliation. By construction, the constraints on $c(\omega)=c$ and $m(\omega)=m$ are satisfied: $c+m \leq g$. Note that each singular leaf by which a pair of tori is pasted together has a unique singularity. Now we will show that the constructed foliation has $h(\ker[\omega])=h$, where $g-m \leq h \leq g$.

If $m \leq 1$, then by Lemma 52 we have $h(\ker[\omega]) = g-m$. So (34) holds and so do the other constraints.

Now let $2 \leq m \leq g$. Consider one cycle $z_j^{(c)}$ in each $T_j^{(c)}$ such that $z_j^{(c)} \in \ker[\omega]$; obviously, the system $\{z_j^{(c)}\}$ is isotropic. We have $h_c \leq h(\ker[\omega])$. If $h_c < h(\ker[\omega])$, then we will complete the system $\{z_j^{(c)}\}$ to a maximal isotropic subgroup $H \subseteq \ker[\omega]$ with $h^{(m)} = h - h_c$ isotropic cycles $z_j^{(m)}$ from $\sharp\, T_i^{(m)}$. Obviously, $0 \leq h^{(m)} \leq m$.

To obtain the desired $h^{(m)}$, we will choose appropriate periods of ω in each $T_i^{(m)}$ without loss of minimality in it.

(i) Let $h^{(m)}=0$. Then just choose all incommensurable periods in all $T_i^{(m)}$.

(ii) Let $h^{(m)}=1$. Choose the periods $(1,\sqrt{2})$ in $T_1^{(m)}$ and $(1,\sqrt{3})$ in $T_2^{(m)}$. Then $\ker[\omega|_{\sharp\, T_i^{(m)}}] = \langle z_{11}-z_{21}\rangle$, where z_{i1}, z_{i2} are the basic cycles of $T_i^{(m)}$ corresponding to these periods. Recall that all other periods are incommensurable.

(iii) Let $h^{(m)}=2$. Similarly, choose the periods $(1,\sqrt{2})$ and $(\sqrt{2},1)$ in the first two $T_i^{(m)}$. Then $\ker[\omega|_{\sharp\, T_i^{(m)}}] = \langle z_{11}-z_{22}, z_{12}-z_{21}\rangle$ is isotropic.

(iv) Let $h^{(m)}=3$. Choose the periods $(1,\sqrt{2})$, $(\sqrt{2},-1)$, and $(\sqrt{2}-1, 2\sqrt{2})$ in the first three $T_i^{(m)}$. By Lemma 14, the isotropic subgroup $\langle z_{11}-z_{21}+z_{31}, z_{12}+z_{22}-z_{31}, z_{12}+z_{21}-z_{32}\rangle$ of $\ker[\omega|_{\sharp\, T_i^{(m)}}]$ is maximal.

(v) Let $h^{(m)} = 2n$, $n \in \mathbb{N}$. Consider n pairs of tori with periods $(\alpha_i, \alpha_i\sqrt{2})$ and $(\alpha_i\sqrt{2}, \alpha_i)$, so that each pair behaves as in (iii) above, but different pairs are incommensurable.

(vi) Let $h^{(m)} = 2n + 1$. Choose $n - 1$ pairs as in (v) and a triple as in (iv).

By construction, we obtain $h^{(m)} + h_c = h(\ker[\omega])$, and $h = h(\ker[\omega])$ satisfies the constraints (14) and (35), i.e., $g - m \le h(\ker[\omega]) \le g$.

Case $k \neq 0$:

Let now $k \ge 2$, thus $g \ge \frac{1}{2}k + 1$.

Figure 11 shows a manifold $M^{(k)}$ of genus $g^{(k)} = \frac{1}{2}k+1$ with $m(\omega^{(k)}) = 1$, $k(\omega^{(k)}) = k$. If $k = 2g-2$, then we have $M^{(k)} = M_g^2$; otherwise we construct a manifold $M^{(0)}$ of genus $g^{(0)} = g - g^{(k)}$ with $m(\omega^{(0)}) = m - 1$ and $k(\omega^{(0)}) = 0$ as discussed above. Then $M^{(k)} \sharp M^{(0)}$ has the desired properties. To obtain $h(\ker[\omega]) = h$, $M^{(0)}$ is to be constructed with $h^{(0)} = \min(h, g^{(0)})$ and in $M^{(k)}$ the periods are constructed as in (i)–(vi) above with $h^{(k)} = h - h^{(0)}$ if positive.

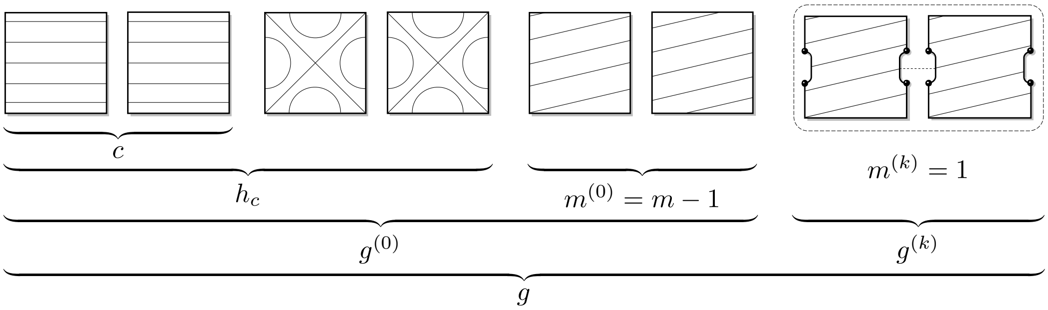

Figure 11: Construction of $M^{(0)} = M(c, h_c, m-1)$ and $M^{(k)}$ in Theorem 55 for the case $k \neq 0$.

If the constraint (33) holds, i.e., $m + c = g - \frac{1}{2}k$, then we have $c = h_c$, so the form can be chosen generic. □

References

[1] S.K. Aranson, E.V. Zhuzhoma, On the structure of quasiminimal sets of foliations on surfaces, Russ. Acad. Sci. Sb. Math. 82 (1995) 397–424.
[2] P. Arnoux, G. Levitt, Sur l'unique ergodicité des 1-formes fermées singulières, Invent. Math. 84 (1986) 141–156.
[3] M. Farber, Topology of Closed One-forms, number 108 in Math. Surv., AMS, 2004.
[4] I. Gelbukh, Presence of minimal components in a Morse form foliation, Differ. Geom. Appl. 22 (2005) 189–198.
[5] I. Gelbukh, Number of minimal components and homologically independent compact leaves for a Morse form foliation, Stud. Sci. Math. Hung. 46 (2009) 547–557.
[6] I. Gelbukh, On the structure of a Morse form foliation, Czech. Math. J. 59 (2009) 207–220.
[7] I. Gelbukh, The number of minimal components and homologically independent compact leaves of a weakly generic Morse form on a closed surface, Rocky Mt. J. Math. (2011) in press.
[8] F. Harary, Graph Theory, Addison-Wesley Publ. Comp., Massachusetts, 1994.
[9] H. Imanishi, On codimension one foliations defined by closed one forms with singularities, J. Math. Kyoto Univ. 19 (1979) 285–291.
[10] V. Jiménez López, G. Soler López, Transitive flows on manifolds, Rev. Mat. Iberoamericana 20 (2004) 107–130.
[11] S. Kono, The structure of quasiminimal sets, Proc. Japan Acad. 46 (1970) 599–604.
[12] G. Levitt, Feuilletages des surfaces, Ann. Inst. Fourier 32 (1982) 179–217.
[13] G. Levitt, 1-formes fermées singulières et groupe fondamental, Invent. Math. 88 (1987) 635–667.
[14] A.G. Maier, Trajectories on closed orientable surfaces, Math. Sbornik 12 (1943) 71–84.
[15] I. Mel'nikova, An indicator of the noncompactness of a foliation on M_g^2, Math. Notes 53 (1993) 356–358.
[16] I. Mel'nikova, A test for non-compactness of the foliation of a Morse form, Russ. Math. Surveys 50 (1995) 444–445.
[17] I. Mel'nikova, Maximal isotropic subspaces of skew-symmetric bilinear mapping, Mosc. Univ. Math. Bull. 54 (1999) 1–3.
[18] A. Zorich, Hamiltonian Flows on Multivalued Hamiltonians on Closed Orientable Surfaces, Preprint, Max-Planck Institut für Mathematik, Bonn, 1994.